*J. Fluid Mech.* (1994), *vol.* 279, *pp.* 165–168



# Note on the inviscid stability of flow over a compliant wall

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(Received 22 November 1993)

This paper is concerned with the linear inviscid stability of parallel flow over a compliant or flexible wall. A Fjørtoft-type criterion providing a necessary condition for instability in terms of the basic velocity field and its second-order derivative is established. This criterion assumes a simple form for basic flows with zero velocity at the wall. For the latter flows, another necessary condition for stability is given. The results are helpful in the search for unstable modes in flow over a compliant wall.

Yeo & Dowling (1987) have shown that the semicircle theorem of Howard (1961) and a result of Høiland (1953) may be extended to parallel inviscid flow over a flexible or compliant wall, albeit in modified form. The semicircle theorem provides a bound for the phase speed $c$ of an unstable mode in the complex $c$-plane, while the result of Høiland gives a bound on its temporal amplification rate. Yeo & Dowling employed a variational Lagrangian representation for the dynamics of the flexible wall. This ensures that their results are applicable to a large class of (passive) compliant walls. Rayleigh's famous inflexion theorem is, however, invalid for compliant walls. Similarly, Fjørtoft's (1950) extension of Rayleigh's theorem is also not valid. In this note, two results concerning the inviscid stability of parallel flow over a compliant wall are derived. They are numbered sequentially after those of Yeo & Dowling (1987) as Propositions 4 and 5. The propositions provide necessary conditions for the linear instability of the flow in terms of the basic velocity profile $U(y)$ and its derivatives. Proposition 4 may be regarded as a form of Fjørtoft criterion for flows over a compliant wall. For the important class of basic flows with zero velocity at the wall ($U_w = 0$), Proposition 4 assumes a particularly simple form. For this class of flow, a further criterion, which involves the velocity gradient at the wall ($U'_w$), is given by Proposition 5. These results reveal the importance of $U_w$ and $U'_w$ in compliant-wall flow stability.

Let $U(y)$ be an inviscid incompressible two-dimensional parallel flow over a compliant wall. A small-amplitude normal-mode perturbation of $U(y)$ has the form $(u,v) = (\partial/\partial y, -\partial/\partial x)\,\phi(y)\exp[\mathrm{i}(\alpha x - \omega t)]$, where $\alpha$ and $\omega$ are the real $x$-wavenumber and the complex frequency of the perturbation respectively. The disturbance function $\phi(y)$ obeys the well-known Rayleigh equation (see Drazin & Reid 1981)

$$\phi'' - \alpha^2\phi - \frac{U''}{U-c}\phi = 0; \tag{1}$$

$c = \omega/\alpha$ is the complex phase speed of the perturbation. We now multiply (1) by $\phi^*$ (superscript * denotes the complex conjugate) and integrate the resultant equation from the mean flow–compliant wall interface at $y = y_w$ to the upper limit of the flow domain at $y_u$; $y_u$ may either be $\infty$ for a semi-bounded flow or a finite $y$-coordinate if the flow is bounded there by a *rigid* surface. For either case, integration by parts yields

$$\int_{y_w}^{y_u}\left(|\phi'|^2+\alpha^2|\phi|^2+\frac{U''(U-c^*)}{|U-c|^2}|\phi|^2\right)\mathrm{d}y = -\phi_w^*\phi_w', \tag{2}$$

where the subscript $w$ indicates evaluation of the function at $y_w$. The upper-limit term $\phi^*\phi'$ at $y_u$ is zero. This is because the disturbance is assumed to decay to zero far away from the wall if the flow domain is semi-infinite; and $\phi(y_u) = 0$ if there is a rigid wall at $y_u$. The surface term $-\phi_w^*\phi_w'$ is governed both by the dynamics of the compliant wall and its interaction with the flow at the mean interface.

For a small-amplitude perturbation, the relevant dynamics of the wall or the surface may be described by a linear relation between the perturbation normal stress $\sigma$ acting on the wall and the resultant vertical displacement $\eta$ of the surface from its original undisturbed position along $y = y_w$. These quantities have the same normal-mode form as the flow perturbation, i.e.

$$\sigma = \hat{\sigma}\,\mathrm{e}^{\mathrm{i}(\alpha x-\omega t)} \quad \text{and} \quad \eta = \hat{\eta}\,\mathrm{e}^{\mathrm{i}(\alpha x-\omega t)},$$

where $\hat{\sigma}$ and $\hat{\eta}$ are the complex amplitudes. A realistic compliant wall necessarily possesses the qualities of elasticity, mass inertia and damping. Yeo & Dowling have shown that for passive compliant walls

$$\hat{\sigma} = (E-\omega^2 I-\mathrm{i}\omega D)\,\hat{\eta}, \tag{3}$$

where $E$, $I$ and $D$ are amplitude-normalized real-valued positive-definite integrals. These integrals are associated with the stored energy (due to elastic behaviour), kinetic energy and dissipation (due to damping) of the perturbation mode in the wall respectively. They are in general functions of $\alpha$, $\omega$ and wall properties. Equation (3) was derived from a variational Lagrangian formulation of the wall dynamics, and is applicable to a large class of compliant walls, including solid-layered walls, plate surfaces and membranes.

The interaction between the flow and the wall is governed by two conditions: the continuity of normal velocity and stress. The linearized form of these conditions is

$$(c-U_w)\,\hat{\eta} = \phi_w, \tag{4a}$$

$$\hat{\sigma} = -\hat{p}_w = (U_w-c)\,\phi_w' - U_w'\,\phi_w, \tag{4b}$$

where $\hat{p}_w$ is the complex amplitude of the flow perturbation pressure $p$ at the mean interface. Using (3) and (4), we have

$$\phi_w^*\phi_w' = [(c^*-U_w)(E-\omega^2 I-\mathrm{i}\omega D)+U_w'\,|U_w-c|^2]\frac{|\hat{\eta}|^2}{(U_w-c)}, \tag{5}$$

which is required for use with (2).

We now multiply (2) by $(\alpha U_w-\omega)$. Taking the imaginary part then yields

$$\omega_i\int_{y_w}^{y_u}\left(|\phi'|^2+\alpha^2|\phi|^2+\frac{U''(U-U_w)}{|U-c|^2}|\phi|^2\right)\mathrm{d}y$$
$$= [-\omega_i E+\omega_i\alpha^2 I(2c_r U_w-|c|^2)+\alpha^2 D(c_r U_w-|c|^2)]\,|\hat{\eta}|^2. \tag{6}$$

The wall-related integrals $E$, $I$ and $D$ on the right-hand side of (6) are all positive. For a *temporally* unstable mode ($\omega_i > 0$), the right-hand side of (6) is negative if the basic flow velocity at the wall $U_w$ and the propagating real phase speed of the mode $c_r$ are of opposite signs. Hence for flow with $U_w \leqslant 0$, the right-hand side of (6) is negative for an unstable mode with $c_r \geqslant 0$. For such an instability mode to exist, the integral on the left-hand side must also be negative. This implies then that the integrand quantity $U''(U-U_w)$ must necessarily be negative in some finite sub-interval(s) of $[y_w, y_u]$. The same condition on the integrand is applicable to unstable upstream-propagating modes ($c_r \leqslant 0$), when the basic flow has wall velocity $U_w \geqslant 0$. These lead us to:

PROPOSITION 4. *For an inviscid parallel flow over a compliant wall with* $U_w \leqslant 0$ ($U_w \geqslant 0$), *a temporally unstable mode with* $c_r \geqslant 0$ ($c_r \leqslant 0$) *can exist only if* $U''(U-U_w) < 0$ *somewhere in the flow.*

Proposition 4 may be regarded as a form of Fjørtoft's theorem. The basic velocity at an inflexion point, $U_s$, in the original Fjørtoft theorem is now replaced by the basic velocity at the wall, $U_w$, which can be seen to play an important role in the new criterion. The criterion is useful mainly for velocity profiles with no-zero $U''$. For a uniform flow, with $U \equiv U_w > 0$ say, Proposition 4 predicts that there are no unstable modes with $c_r < 0$, which is in agreement with the bound given by the modified semicircle theorem of Yeo & Dowling (1987). Proposition 4 has the following simple consequence for basic flows with $U_w = 0$:

COROLLARY. *A necessary condition for an inviscid parallel flow over a compliant wall with* $U_w = 0$ *to be unstable is that* $U''U < 0$ *somewhere in the flow.*

This is a particularly important case because all real basic flows over a wall, be it compliant or rigid, necessarily have $U_w = 0$. The corollary is therefore relevant to the inviscid instability of real flows. Cases with $U_w \neq 0$ are merely idealizations which substitute a thin shear layer by a vortex sheet for simplicity of analysis.

For flows with $U(y) > 0$, except at the wall where $U_w = 0$, the corollary implies that $U'' < 0$ somewhere for an instability to exist. An example is given by the Blasius velocity profile, which is known to be unstable when the wall becomes sufficiently compliant. The converse equivalent of the corollary states that a sufficient condition for a flow ($U_w = 0$) to be *linearly* stable is that $UU'' \geqslant 0$ everywhere in the flow. Thus, a plane Couette flow over a compliant wall with $U_w = 0$ is linearly stable since it has $U'' \equiv 0$ everywhere. We may further note that a flow which is inviscidly stable over a compliant wall will remain stable when the wall is rigid. This is because a linearly stable flow over a compliant wall must have $UU'' \geqslant 0$ everywhere. This condition generally excludes the occurrence of an inflexion point within the flow (the exceptions being flows with an inflexion point in the basic velocity profile occurring at an interior point where $U$ is also zero and such that the product $UU'' \geqslant 0$ across the point). The occurrence of inflexion point(s) in the velocity profile is a necessary condition for the inviscid instability of the flow over a rigid wall, according to Rayleigh's inflexion theorem.

A further result for compliant walls may be deduced by considering the imaginary part of (2):

$$\omega_i \int_{y_w}^{y_u} \frac{U''}{|U-c|^2} |\phi|^2 \, \mathrm{d}y = -\omega_i (2Ec_r + |c|^2 U'_w) \frac{|\hat{\eta}|^2}{|c|^2} - c_r \alpha^2 D |\hat{\eta}|^2, \tag{7}$$

where the flow is again assumed to have $U_w = 0$. For a flow which has $U'_w \geqslant 0$, the right-hand side of (7) is negative for unstable modes with positive phase speed ($c_r > 0$). Such an instability can exist only if $U'' < 0$ over some finite sub-interval(s) in $[y_w, y_u]$. An equivalent result is applicable for cases with $U'_w \leqslant 0$. Hence:

PROPOSITION 5. *For an inviscid flow with* $U_w = 0$ *and* $U'_w \geqslant 0$ ($U'_w \leqslant 0$) *over a compliant wall, an unstable mode with* $c_r > 0$ ($c_r < 0$) *can exist only if* $U'' < 0$ ($U'' > 0$) *somewhere in the flow.*

This result indicates that for flows over a compliant wall with $U_w = 0$ and $U'_w \geqslant 0$, the instability wave modes with $c_r > 0$ are in some sense related to the existence of negative $U''$. When $U_w \neq 0$, the right-hand side of (7) becomes highly complex and no useful deductions have been forthcoming. The complexity of this latter case may be attributed to the fact that an idealized vortex sheet at a *deformable* surface may be unstable on its own merit quite independently of the existence of non-zero $U''$ elsewhere in the flow. The stability or instability of this vortex sheet is closely coupled to the specific properties of the wall, which partially explains the absence of a simple result for the $U_w \neq 0$ case. Again we should emphasize that the wall vortex sheet is in reality a very thin shear layer with $U_w = 0$, whose instability, if any, is related to non-zero $U''$, subject to the preceding corollary and Proposition 5.

The magnitude of wall displacement $|\hat{\eta}|$ tends to zero in the limit of an infinitely stiff (rigid) wall. For any finite normalization of the flow perturbation, $E|\hat{\eta}|$, $I|\hat{\eta}|$, $D|\hat{\eta}|$ are at most of $O(1)$ according to (3). The terms on the right-hand sides of (6) and (7) are then of $O(|\hat{\eta}|)$ and thus vanish for a rigid wall. Equation (7) then yields Rayleigh's inflexion theorem. In the absence of a perturbation interaction between the flow and a rigid wall, the velocity $U_w$ on the left-hand side of (6) need no longer have to be the flow velocity at the wall, and may be a freely selected real number. If $U_w$ is replaced by $U_s$ (the velocity at a point of inflexion), then Fjørtoft's extension of Rayleigh's theorem is also obtained (see Drazin & Reid 1981).

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